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A Fano 3-fold with the 1-dimensional locus of non-rational singularities

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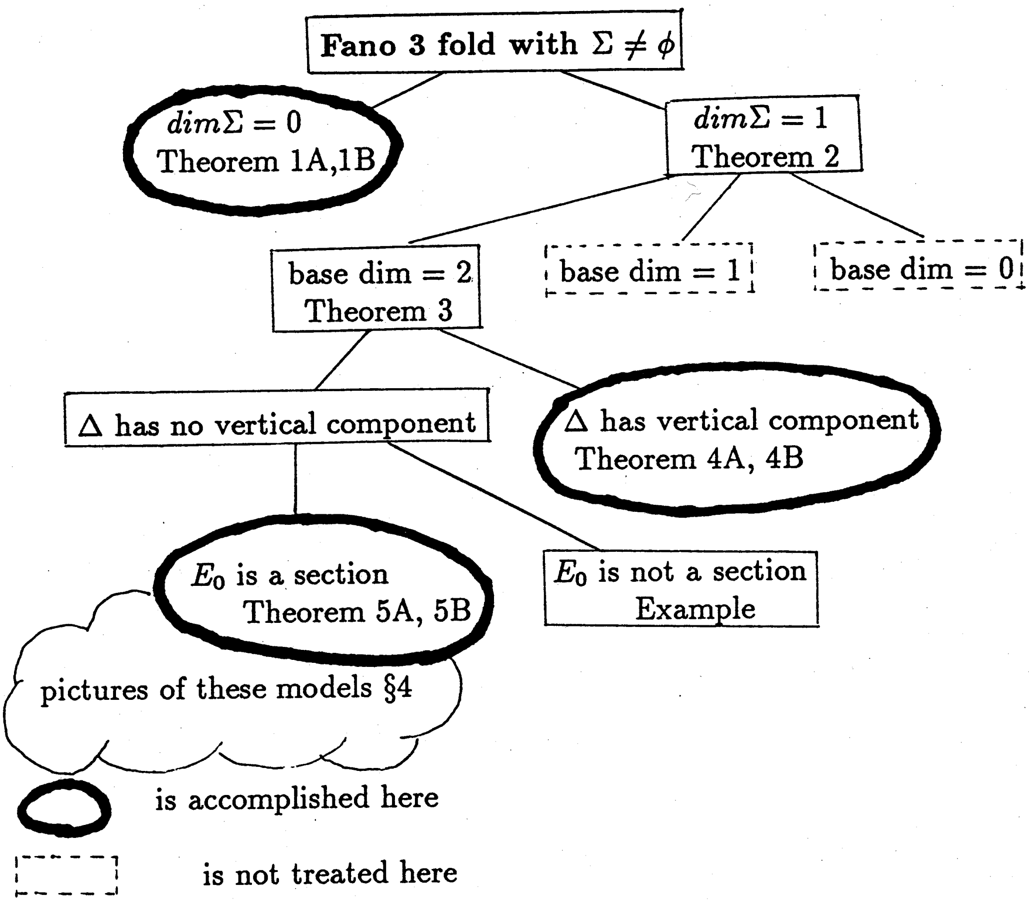
The structure of this paper



Introduction

In this paper a Fano 3-fold means a normal projective variety of dimension three over **C** whose anticanonical sheaf is ample and invertible. During the past fifteen years, there has been big progress in the investigation of a non-singular Fano 3-fold owing to Iskovskih, Mori, Mukai and Shokulov. And it is still developing. On the other hand, in singular Fano 3-folds, progress seems to have started recently. Here we study the structure of a Fano 3-fold with non-rational singularities.

Let Σ be the locus of non-rational singular points of a Fano 3-fold X. As X is normal, $dim\Sigma \leq 1$. If $dim\Sigma = 0$, then X is isomorphic to a projective cone over a normal K3-surface or an Abelian surface (Theorem 1A, 1B). The proof of this theorem also works in the case that Σ contains an isolated point. So what we should study next is the case that Σ has pure dimension one. Such a Fano 3-fold is classified in three families according to the maximal basis-dimension of its Q-factorial terminal modification (Theorem 2, Definition 1). We obtain the fact that a Fano 3-fold with the maximal basis-dimension 2 admits a projective bundle over a non-singular surface as a **Q**-factorial terminal modification (Theorem 3). We try to make clear the stucture of a Fano 3-fold in this family: what kind of surface occurs as a basis, what kind of projective bundle appears as a **Q**-factorial terminal modification and which parts on the projective bundle are contracted in a Fano 3-fold.

The author would like to thank Professors Nakayama and Kei-ichi Watanabe and also other members of Waseda Seminar for their stimulating discussion during the preparation of this article. In particular Nakayama's proof of Proposition 2 helped her very much and also K-i. Watanabe's comment "a Weil divisor on a Q-factorial terminal singularity is Cohen-Macaulay" was very helpful in the proof of Theorem 2.

§1. The case $dim\Sigma = 0$

Theorem 1A([I]). Let X be a Fano 3-fold with $dim\Sigma = 0$. Then there exist a normal surface S which is either an Abelian surface or a normal K3-surface and an ample invertible sheaf $\mathcal{L}$ on S such that X is the contraction of the negative section of a projective bundle $\mathbf{P}(\mathcal{O}_S \oplus \mathcal{L})$. Here a normal K3-surface implies a normal projective surface with the trivial canonical sheaf and has only rational singularities.

Theorem 1B([I]). Let X be a projective cone over a surface S which is either an Abelian or a normal K3-surface. Then X is a Fano 3-fold with $\Sigma = \{\textit{the vertex}\}$.

§2. Basic structure theorem of Q-factorial terminal modifications for the case $dim\Sigma = 1$

Theorem 2. Let X be a Fano 3-fold with Σ of pure dimension one. Let $g : Y \to X$ be a **Q**-factorial terminal modification whose existence is proved by Mori ([M]). Denote $K_Y = g^*K_X - \Delta$. Then we have a sequence of projective morphisms:

$Y = Y_0 \xrightarrow{\varphi_0} Y_1 \xrightarrow{\varphi_1} Y_2 . . \longrightarrow Y_r \xrightarrow{\varphi_r} Z$, where for each i, φ_i is the contraction of an extremal ray R_i on Y_i such that $R_i\Delta_i > 0$ (here, $\Delta_0 = \Delta$, and $\Delta_i = (\varphi_{i-1})_*\Delta_{i-1}$). For $i \leq r-1$, φ_i is a birational contraction of a divisor isomorphic to $F_{a,0}$ ($a \geq 1$) to a non-singular point and φ_r is a fibration to a lower dimensional variety Z.

Definition 1. The variety Z above is called a basis of X. And each φ_i is called a Δ-extremal contraction. Of course a basis of X is not unique for X. It depends on the choice of a **Q**-factorial terminal modification Y and also on the choice of extremal rays R_i's.

From now on, we devote to study X which has a two dimensional basis Z. In this case, the last contraction $\varphi_r : Y_r \to Z$ satisfies the assumption of the following proposition. So we can see that it is a $\mathbf{P}^1$-bundle over a non-singular surface Z.

Proposition 1 (Nakayama). Let $\varphi : Y \to Z$ be a contraction of an extremal ray on a 3-fold Y with at worst **Q**- factorial terminal singularities on it to a surface Z. Assume there exists an invertible sheaf on Y whose degree on a general fiber is 1. Then Z is non-singular and Y is a $\mathbf{P}^1$-bundle over Z.

Theorem 3. Let X be a Fano 3-fold with one dimensional Σ and a two dimensional basis. Then there exists a **Q**-factorial terminal modification $g : Y \to X$ such that a Δ-extremal contraction $\varphi_0 : Y \to Z$ gives a $\mathbf{P}^1$-bundle over a non-singular surface Z.

This theorem is proved by applying the following lemma successively.

Lemma. Let X be as above and $Y = Y_0 \xrightarrow{\varphi_0} Y_1 \xrightarrow{\varphi_1} Y_2 . . \longrightarrow Y_r \xrightarrow{\varphi_r} Z$ be a sequence of Δ-extremal contractions of **Q**-factorial terminal modification Y of X with 2-dimensional basis Z. If $r > 0$, then there is a flop Y_i' of Y_i for each i ($i \leq r-1$) such that $g' : Y' = Y_0' \to X$ is a **Q**-factorial terminal modification of X and $Y' = Y_0' \xrightarrow{\varphi_0'} Y_1' \xrightarrow{\varphi_1'} Y_2' . . \longrightarrow Y_{r-1}' \xrightarrow{\varphi_{r-1}'} Z'$ is a sequence of Δ'-extremal contractioins with 2-dimensional basis Z', where Δ' is a **Q**-divisor such that $K_{Y'} = g'^*K_X - \Delta'$.

§3. Fano 3-folds which have $\mathbf{P}^1$-bundles as Q-factorial terminal modi-

fications.

Let X be a Fano 3-fold with a 2-dimensional basis. Then, by Theorem 3, we can take a $\mathbf{Q}$-factorial terminal modification $g : Y \to X$ such that a Δ-extremal contraction $\varphi : Y \to Z$ gives a $\mathbf{P}^1$ -bundle over a non-singular surface Z. Then we have the following facts:

(i) $-g^*K_X\ell = \Delta\ell = 1$, where ℓ is a fiber of $\varphi : Y \to Z$.

(ii) Δ is denoted by $E_0 + \varphi^*(\Delta')$, where E_0 is an irreducible component with $E_0\ell = 1$ and $\Delta' \in Pic(Z)$.

The case $Supp\Delta$ **contains a vertical component**

We call an irreducible divisor D in Y a vertical divisor for g, if D is mapped to a point of X by g.

Theorem 4A. Let $X, g : Y \to X, \Delta$ and $\varphi : Y \to Z$ be as in the beginning of this section. Asssume $Supp\Delta$ contains a vertical component.

Then, (i) a vertical component is unique and coincides with E_0 and it is a section of the projection φ,

(ii) there exists a normal surface Z_0 with at least one non-rational singular point on it whose canonical sheaf is trivial and whose minimal resolution is $h : Z \to Z_0$ and

(iii) the $\mathbf{P}^1$-bundle $\varphi : Y \to Z$ is a pull back of a $\mathbf{P}^1$-bundle $\varphi_0 : Y_0 \to Z_0$ by h and $g : Y \to X$ factors as $Y \xrightarrow{h} Y_0 \xrightarrow{g_0} X$, where g_0 is a contraction of the negative section $h(E_0)$.

Theorem 4B. Let S be a normal surface with trivial canonical sheaf and at least one non-rational singular point on it. Then an arbitrary projective cone X over S is a Fano 3-fold and Σ is generating lines over a non-rational singular points of S.

Remark. Normal surfaces with the trivial canonical sheaf and at least one non-rational singular point are studied in [U] among others. The number of non-rational singular points is less than or equall to 2. It is 2, if and only if both of them are simple elliptic singularities [U, Theorem 1].

The case $Supp\Delta$ **contains no vertical component**

In the previous case, E_0 is a section of φ. But in this case, it is not necessarily true. First we consider the case that E_0 is a section. Since E_0 is not a vertical component, $g|_{E_0} : E_0 \to C$ is a fibration to a curve C.

Proposition 2. The possible triples $(E_0, g|_{E_0}, \Delta')$ are the following:
(i) $(\mathbf{P}^1 \times$ *elliptic curve, the first projection* $p_1, \phi)$,
(ii) (a rational elliptic surface, the elliptic fibration, ϕ),
(iii) E_0 is the composite of r-blowing ups $E_0 \xrightarrow{\sigma_r} .. \xrightarrow{\sigma_1} \mathbf{P}^1 \times$ *elliptic curve*, where σ_1 is the blow up at a point on the fiber $C = p_1^{-1}(z)$ of a point $z \in \mathbf{P}^1$ and σ_i $(i > 1)$ is the blow up at the intersection of the proper transform of C and the exceptional curve of σ_{i-1}. The morphism $g|_{E_0}$ is $p_1\sigma_1\sigma_2..\sigma_r$ and Δ' =the proper transform of C.
(iv) E_0 is a ruled surface $p : E_0 \to S$ such that there exist a covering $\pi : S \to \mathbf{P}^1$ and a member D in $|-K_{E_0}|$ of type $D = (\pi p)^*(z) + \Delta'$, where $z \in \mathbf{P}^1$ and Δ' is an effective divisor with $K_{E_0}C \geq 0$ for every component $C \subset \Delta'$. The morphism $g|_{E_0}$ is πp.

Theorem 5A. Let $X, g : Y \to X, \Delta, \Delta'$ and $\varphi : Y \to Z$ be as in the beginning of this section. Assume $Supp\Delta$ contains no vertical component and E_0 is a section of φ. Denote $-g^*K_X = E_0 + \varphi^*L$ for $L \in PicZ$. Then the triple $(E_0, g|_{E_0}, \Delta')$ is as one of (i)$\sim$(iv) in Proposition 2 and the $\mathbf{P}^1$-bundle $\varphi : Y \to Z$ is obtained by an sheaf $\mathcal{E}$ which satisfies the following properties:
(I)$\mathcal{E}$ is an extension of $\mathcal{N}$ by $\mathcal{O}_Z$, where $\mathcal{N} = \mathcal{O}_Z(-K_Z - \Delta' - L)$ such that $\mathcal{E}|_{\Delta'} = \mathcal{O}_{\Delta'}(-L) \oplus \mathcal{O}_{\Delta'}(-L)$ and $(L \otimes \mathcal{E})_y$ is generated by its global sections for each $y \in \Delta'$.
(II) $L - \Delta'$ is semi-ample and $(L - \Delta')(L - \Delta' - K_Z) > 0$.

Theorem 5B. Let a triple $(Z, \tilde{g}, \Delta')$ be as one of (i) $\sim$ (iv) in Proposition 2 and $L \in PicZ$ and $\mathcal{E}$ be as in (I) and (II) in Theorem 5A.
Let $Y = \mathbf{P}(\mathcal{E}) \xrightarrow{\varphi} Z$ be the projective bundle defined by $\mathcal{E}$ and E_0 be a section of φ defined by the surjection $\mathcal{E} \to \mathcal{N}$. Denote $E_0 + \varphi^*L$ by H.
Then $|mH|$ is base point free for $m \gg 0$ and the image X of the morphism $g = \Phi_{|mH|} : Y \to \mathbf{P}^M$ becomes a Fano 3-fold with one dimensional Σ and $g|_{E_0} = \tilde{g}$ under the identification of E_0 with Z.

Now we give an example of a Fano 3-fold with E_0 not a section.

Example. Let Z be the projective plane $\mathbf{P}^2$, C and C' be two general curves of degree 3 on Z. Let $\sigma : \tilde{Z} \to Z$ be the blowing up at 9-distinct points $\{p_1, p_2, .., p_9\} = C \cap C'$, then $\tilde{Z}$ becomes an elliptic surface with elliptic fibers $[C]$, $[C']$, where $[C]$ is the proper transform of C on $\tilde{Z}$. Denote the fiber $\sigma^{-1}(p_i)$ by ℓ_i. Let L be $\sigma^* L_0 + \sum_{i=1}^{9} \ell_i$ where L_0 is an ample divisor on Z.

Since $H^1(\tilde{Z}, L-[C]) \simeq \oplus H^1(\ell_i, L-[C]|_{\ell_i}) \simeq \mathbf{C}^{\oplus 9}$, we can take an extension sheaf $\tilde{\mathcal{E}}$ of $\mathcal{O}([C]-L)$ by $\mathcal{O}_{\tilde{Z}}$ such that the restriction $[\tilde{\mathcal{E}}|_{\ell_i}] \in H^1(\ell_i, L-[C]|_{\ell_i})$ is not zero for every i ($i = 1, 2, .., 9$). Now $0 \to \mathcal{O}_{\tilde{Z}}|_{\ell_i} \to \tilde{\mathcal{E}}|_{\ell_i} \to \mathcal{O}([C]-L)|_{\ell_i} = \mathcal{O}_{\mathbf{P}^1}(2) \to 0$ does not split, so $\tilde{\mathcal{E}}|_{\ell_i} \simeq \mathcal{O}_{\mathbf{P}^1}(1) \oplus \mathcal{O}_{\mathbf{P}^1}(1)$. Put $\tilde{\mathcal{E}}' = \tilde{\mathcal{E}}(\Sigma \ell_i)$, then $\tilde{\mathcal{E}}'|_{\ell_i}$ is trivial for each i. By Schwarzenberger's Theorem, $\tilde{\mathcal{E}}' = \sigma^* \mathcal{E}$ for some locally free sheaf $\mathcal{E}$ on Z. Let Y be the projective bundle $\mathbf{P}(\mathcal{E})$ and $\tilde{Y}$ be $\mathbf{P}(\tilde{\mathcal{E}}')$. Then we have the diagram of a fiber product

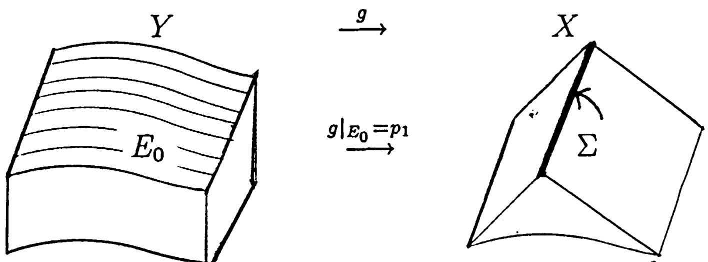
$$\begin{array}{ccc} \tilde{Y} & \xrightarrow{\sigma} & Y \\ \downarrow \tilde{\varphi} & \square & \varphi \downarrow \\ \tilde{Z} & \xrightarrow{\sigma} & Z \end{array}$$

Let $\tilde{E}_0$ be the section of $\tilde{\varphi}$ defined by the surjection $\tilde{\mathcal{E}} \to \mathcal{O}([C]-L)$ and E_0 be the image $\sigma(\tilde{E}_0)$. Then $H = E_0 + \varphi^* L_0$ is a semipositive divisor on Y. The image X of the morphism $\Phi_{|mH|} : Y \to \mathbf{P}^M$ becomes a Fano 3-fold with $\Sigma \simeq \mathbf{P}^1$ and Y is a $\mathbf{Q}$-factorial terminal modification of X. It is easy to see that $\Delta = E_0$ and E_0 contains the fibers of φ over $p_1, p_2, .., p_9 \in Z$.

§4. Pictures of Y and X of Theorem 5

(i)In the case the triple is $(\mathbf{P}^1 \times C$, the first projection $p_1, \phi)$, where C is an elliptic curve. Then $\Delta = E_0$. If we denote $L = p_1^* \mathcal{O}_{\mathbf{P}^1}(a) \otimes p_2^* B$, then $a \geq 0$ and B is ample.

(i-1) $a > 0$. $g|_{Y-E_0} : Y - E_0 \simeq X - \Sigma$.

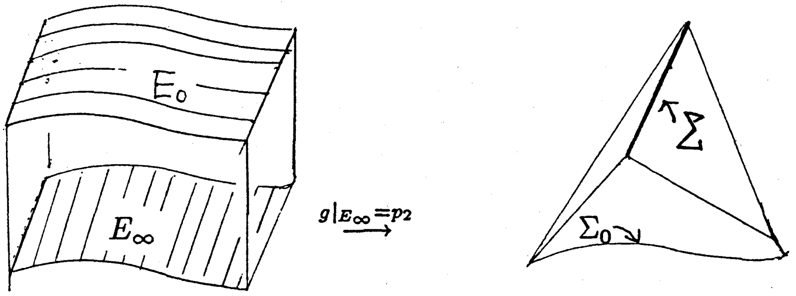


(i-2) $a = 0$ and the exact sequence $(\mathcal{E}) : 0 \to \mathcal{O}_Z \to \mathcal{E} \to \mathcal{N} \to 0$ splits.

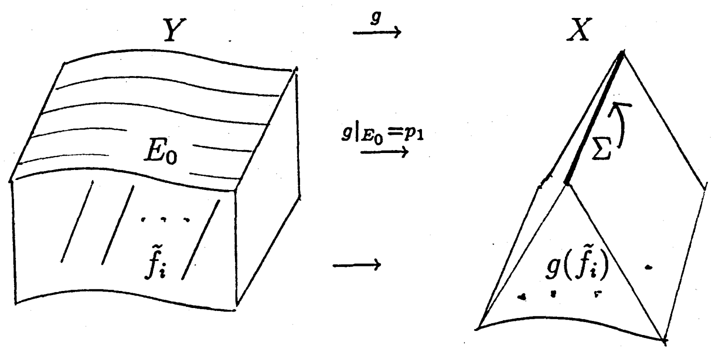
$g|_{Y-E_0-E_\infty} : Y - E_0 - E_\infty \simeq X - \Sigma - \Sigma_0$, and $g|_{E_\infty} = p_2$, where Σ_0 is the locus of canonical singularities.

$Y \qquad \xrightarrow{g} \qquad X$

$E_0 \qquad \xrightarrow{g|_{E_0}=p_1} \qquad \Sigma$

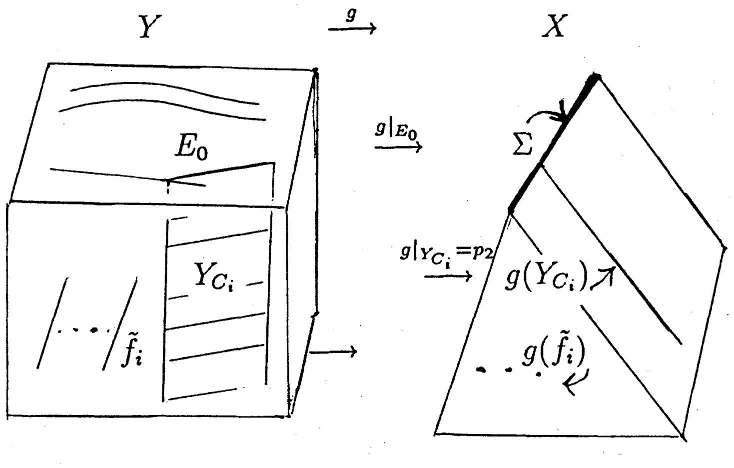


(i-3) $a = 0$ and the exact sequence $(\mathcal{E}) : 0 \to \mathcal{O}_Z \to \mathcal{E} \to \mathcal{N} \to 0$ does not split. There exists a divisor $\sum_{i=1}^{s} m_i q_i \in |B|$ such that the restriction $(\mathcal{E})|_{f_i}$ splits for each i,$(i = 1,..,s)$, where $f_i = p_2^{-1}(q_i)$. For a general fiber $f = p_2^{-1}(q), q \in C$, Y_f is $\mathbf{P}^1 \times \mathbf{P}^1$ and for f_i, $Y_{f_i} \simeq \mathbf{P}(\mathcal{O} \oplus \mathcal{O}(2))$. $E_0|_{Y_f}$ is an ample section for general f and is the disjoint section from the negative section for $f = f_i$. Denote the negative section of Y_{f_i} by $\tilde{f}_i$. Then the restriction $g|_{Y-E_0-\cup \tilde{f}_i}$ is an isomorphism, $g|_{E_0} = p_1$, and each $\tilde{f}_i$ is contracted to a canonical singularity in X.

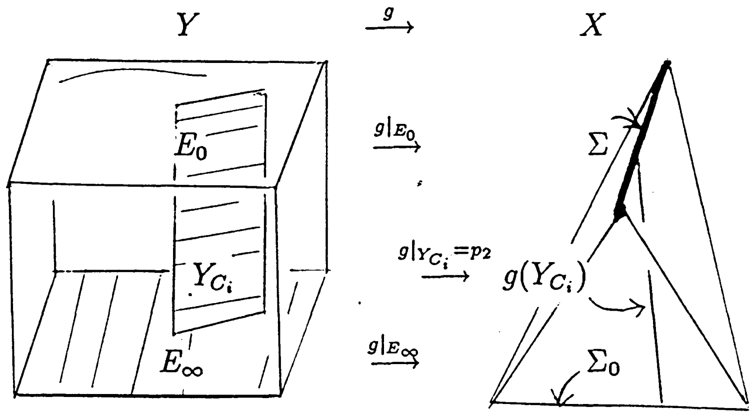


(ii)The case that the triple $(E_0, g|_{E_0}, \Delta')$ is (rational elliptic surface, the elliptic fibration, ϕ). Then $E_0 = \Delta$ in this case too. If L is big then the exact sequence $(\mathcal{E})$ splits and if L is not big $|L|$ gives a fibration $\Phi = \Phi_{|L|} : Z \to \mathbf{P}^1$ with a general fiber $\mathbf{P}^1$. Let C_i $(i = 1, 2, .., r)$ be (-2)-curves on Z with $LC_i = 0$ and f_j $(j = 1, .., s)$ be (-1)-curves on Z with $Lf_i = 0$. Then $E_0|_{Y_{f_i}}$ is the section disjoint from the negative section. Denote the negative section of Y_{f_i} by $\tilde{f}_j$. Then the normal bundle of $\tilde{f}_j$ in Y is $\mathcal{O}_{\mathbf{P}^1}(1) \oplus \mathcal{O}_{\mathbf{P}^1}(1)$.

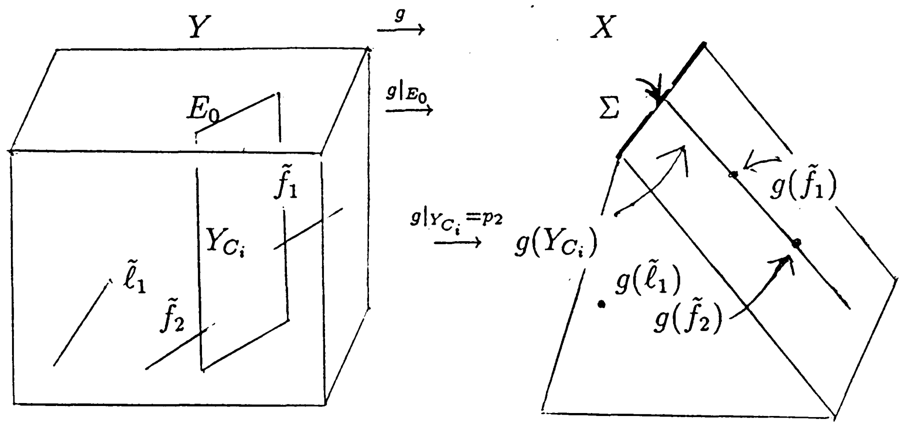
(ii-1) L is big. Then the restriction $g|_{Y-E_0-\cup Y_{C_i}-\cup \tilde{f}_i}$ is an isomorphism, $g|_{Y_{C_i}}$: $Y_{C_i} \simeq C_i \times \mathbf{P}^1 \to \mathbf{P}^1$ is the projection to the second factor and $g(\tilde{f}_j)$ is an isolated canonical singular point for each j. A point of $g(Y_{C_i})$ away from $g(E_0)$ is non-isolated canonical singularities.



(ii-2) L is not big and $(\mathcal{E})$ splits. Let E_∞ be the section of φ disjoint from E_0. Then the restriction $g|_{E_0-E_\infty-\cup Y_{C_i}}$ is an isomorphism, $g|_{Y_{C_i}}$ is as above and $g|_{E_\infty}=\Phi$.

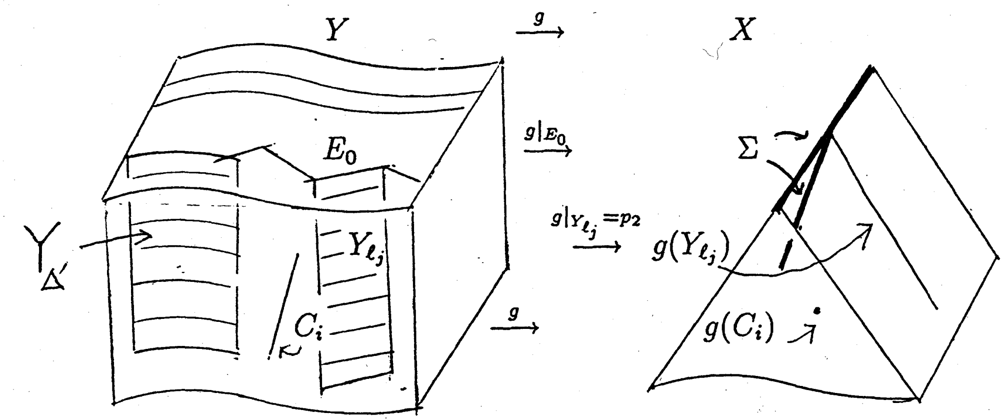


(ii-3) L is not big and $(\mathcal{E})$ does not split. Denote $L=\Phi^*L_0$ for a Cartier divisor L_0 on $\mathbf{P}^1$ Then the extension $\mathcal{E}$ of $\mathcal{N}$ corresponds to a non-zero section $\phi_{\mathcal{E}}$ of $\Gamma(\mathbf{P}^1, L_0+K_{\mathbf{P}^1})$. Let $\phi_{\mathcal{E}}$ define a divisor $\sum_{k=1}^{d} m_k q_k$ $(d\geq 0, m_k>0)$ and ℓ_k $k=1,..,b$ $(0\leq b\leq d)$ be smooth fibers among $\{\Phi^{-1}(q_k)\}$. A component of a singular fiber of Φ is either one of C_i's or f_i's defined above. For a general fiber $\ell=\Phi^{-1}(q)$ $q\in\mathbf{P}^1$, Y_ℓ is $\mathbf{P}^1\times\mathbf{P}^1$ and for ℓ_k, $Y_{\ell_k}\simeq\mathbf{P}(\mathcal{O}\oplus\mathcal{O}(2))$. $E_0|Y_\ell$ is an ample section for general ℓ, while it is the section disjoint from the negative section for $\ell=\ell_k(0\leq k\leq b)$. Denote the negative section of Y_{ℓ_k} by $\tilde{\ell}_k$. Then the restriction $g|_{Y-E_0-\cup_{i=1}^{r}Y_{C_i}-\cup_{j=1}^{s}\tilde{f}_j-\cup_{k=1}^{b}\tilde{\ell}_k}$ is isomorphic, $g|_{Y_{C_i}}$ is the second projection, $\tilde{f}_j'$s and $\tilde{\ell}_k'$s are contracted to canonical singularities in X.



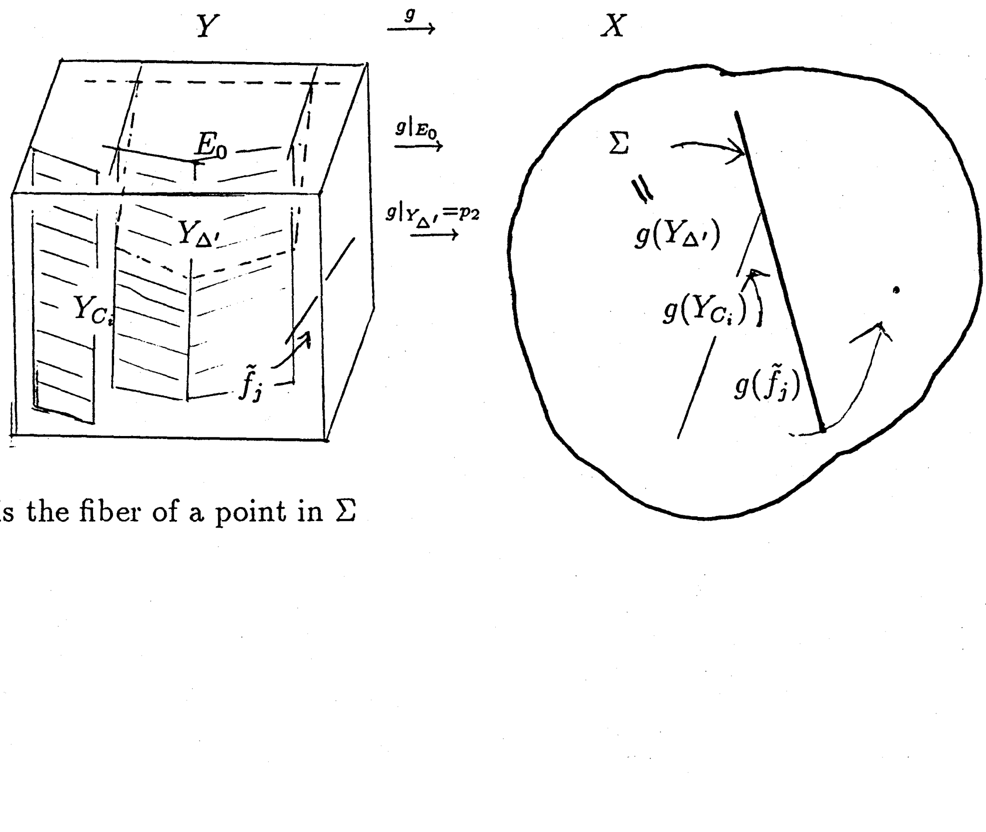
(iii) The case that the triple $(E_0, g|_{E_0}, \Delta')$ is as follows: E_0 is the composite of r-blowing ups $E_0\xrightarrow{\sigma_r}..\xrightarrow{\sigma_1}\mathbf{P}^1\times$ *elliptic curve*, where σ_1 is the blow up at a point on the fiber $C=p_1^{-1}(z)$ of a point $z\in\mathbf{P}^1$ and σ_i $(i>1)$ is the blow up at the intersection of the proper transform of C and the exceptional curve of σ_{i-1}. The morphism $g|_{E_0}$ is $p_1\sigma_1\sigma_2..\sigma_r$ and Δ' =the proper transform of C.

Then L is nef and big, with $L\ell_r > 0$ and the exact sequence $(\mathcal{E})$ splits, where ℓ_i $(i = 1, 2, .., r)$ are the exceptional curves of σ_i respectively. Let E_∞ be the section of φ disjoint from E_0, and ℓ_j $j \in J \subset \{1, 2, .., r-1\}$ be the exceptional curves with $L\ell_j = 0$ and C_i $i = 1, 2, ..s$ be the (-1) curves on E_∞ with $LC_i = 0$. Then $g|_{Y-E_0-Y_{\Delta'}-\cup_{j\in J}Y_{\ell_j}-\cup_{i=1}^{n}C_i}$ is an isomorphism, $g|_{Y_{\ell_j}} \simeq \ell_j \times \mathbf{P}^1 \to \mathbf{P}^1$ is the projection to the second factor and $g(C_i)$ is an isolated canonical singular point on X for $i = 1, 2, .., s$.



(iv) The case that the triple is as (ii) of Proposition 2.

Then the exact sequence $(\mathcal{E})$ does not split. Let C_i $(i = 1, 2, .., r)$ be (-2)-curves on Z with $eC_i = LC_i = 0$ and f_j $(j = 1, .., s)$ be (-1)-curves on Z with $ef_j > 0$ and $Lf_j = 0$ Then we can take the negative section $\tilde{f}_j$ of $Y_{\tilde{f}_j}$ disjoint from E_0. Then $g|_{Y-\Delta-\cup_{i=1}^{r}Y_{C_i}-\cup_{j=1}^{s}\tilde{f}_j}$ is an isomorphism, $g|_{Y_C} : Y_C \simeq C \times \mathbf{P}^1 \to \mathbf{P}^1$ is the projection to the second factor for a component $C < \Delta$ and for $C = C_i$ $(i = 1, .., r)$ and $g(\tilde{f}_j)$ is an isolated canonical singularity for $j = 1, .., s$.



,where ..... is the fiber of a point in Σ

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